Clifford Residues and Charge Quantization

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February 7, 2008

Abstract

We derive the quantization of action, particle number, and *electric* charge in a Lagrangian spin bundle over $\mathbb{M} \equiv \mathbb{M}_{\#} \backslash \cup D_J$, Penrose's conformal compactification of Minkowsky space, with the world tubes of massive particles removed.

Our Lagrangian density, $\mathcal{L}_g$, is the spinor factorization of the Maurer-Cartan 4-form Ω^4; it's action, S_g, measures the covering number of the 4 *internal* $u(1) \times su(2)$ phases over external spacetime $\mathbb{M}$. Under PTC symmetry, $\mathcal{L}_g$ reduces to the second Chern form $TrK_L \wedge K_R$ for a left $\oplus$ right chirality spin bundle. We prove a *residue theorem* for $gl(2, \mathbb{C})$-valued forms, which says that, when we "sew in" singular loci D_J over which the $u(1) \times su(2)$ phases of the matter fields have some extra twists compared to the **8** vacuum modes, the additional contributions to the action, electric charge, lepton and baryon numbers are all *topologically quantized*. Because left and right chirality 2-forms are *chiral dual,* forms are quantized over their *dual* cycles. Thus it is the interaction $c_2(E)$, with a globally nontrivial *magnetic* field, that forces *electric fields* to be topologically quantized over *spatial* 2 cycles, $\int_{\mathbb{S}^2} K_{or} e^{\theta} \wedge e^{\varphi} = 4\pi N$.

1 Introduction

Yang-Mills monopoles [1] have topologically quantized *magnetic* charges because it is the *magnetic* parts, $K_{jk} e^j \wedge e^k$ $(j, k = 1, 2, 3)$, of their $su(2)$-valued spin-curvature 2 forms, $K = \mathbf{d}\Omega + \Omega \wedge \Omega$, that "wrap" integrally around *spatial* 2-cycles. For electric charge to be topologically quantized, the *electric* field would have to wrap integrally about its *dual (spatial)* cycle,

$$\int_{\mathcal{S}_2} K_{0r} e^{\theta} \wedge e^{\varphi} = 4\pi N;$$

Gauss' law.

The *instantons* and *dyons* of Yang Mills theories, which possess (anti-) *self-dual* curvatures, also possess nonzero electric fields, $K_{0j} = \pm\epsilon_j^{k\ell} K_{k\ell}$, and electric charges that are quantized because their magnetic charges are. However, they live in a *Euclidean* four-space. Any analogous construction in Minkowsky space, where $** = -1$, must have imaginary (anti-) self-dual curvatures $*K = \pm iK$.

We exhibit a model here in which Left and Right spin curvatures are *imaginary chiral dual*: $K_R = \pm i * K_L$. Localized chiral dual solutions are dyons with half-integral units of electric and magnetic charges. For PT *antisymmetric* (PT_A) solutions, the electric semi-charges add, while the magnetic semi-charges cancel, thus binding together the left and right chiral halves into a bispinor particle.

In the work of Van der Waerden [2], Sachs [3], Penrose [4], and Keller [5], [6], it becomes clear that geometric and Fermionic fields are the integral and half-integral sectors of one unified spin-4 tensor field.

In a companion paper [7] (see also [8]), we exhibited a grand-unified Lagrangian density,

$$\mathcal{L}_g = i \int_{\mathbb{M}} \mathbf{d}\varsigma^{\pm}\xi_{\mp} \wedge \chi^{\pm}\mathbf{d}\eta_{\pm} \wedge \mathbf{d}\chi^{\mp}\eta_{\pm} \wedge \varsigma^{\mp}\mathbf{d}\xi_{\pm}, \tag{1}$$

(*sum over all neutral sign combinations*) invariant under the group E_P of *passive* Einstein transformations; Sachs' [3] term for the global extension of the Poincaré group to a Friedmann universe. E_P transformations connect the *same* physical state in the moving frames of different observers. In the PTC-symmetric *geometrical optics* (*g.o.*) regime in $\mathbb{M} = \mathbb{M}_{\#} \backslash \cup D_J$, outside the *singular loci* D_J, $\mathcal{L}_g$ reduces to the *Maurer-Cartan* 4 *form.* This gives a natural topological action

$$S_g = i \int_{\mathbb{M}} Tr\Omega^L \wedge \Omega_L \wedge \Omega^R \wedge \Omega_R \equiv i \int_{\mathbb{M}} \widehat{\mathcal{L}}_g, \tag{2}$$

which measures the covering number of spin space over spacetime, and comes in quantized units.

$\mathcal{L}_g$ of (1) is not unique—but its action S_g (2) does have a desirable feature: The terms in S_g decompose into effective *electroweak, strong,* and *gravitational* potentials and curvatures, together with their proper field actions [7]. We show here, using *spin residues*—"winding numbers" of $gl\,(2,\mathbb{C})$-valued forms about each codimension J, singularity D_J—that these *actions* and charges are *topologically quantized.*

The *singular loci* D_J are where $J = 1, 2, 3$, or 4 pairs of spin rays cross, forming *caustics.* Here the $gl\,(2,\mathbb{C})$ *phases* of J chiral pairs of spinors, i.e. the local, path-dependent exponents in the geometrical optics (*g.o.*) ansatz

$$\psi\,(x) = e^{\frac{i}{2}(\theta^{\alpha}(x)+i\varphi^{\alpha}(x))\sigma_{\alpha}}\psi\,(0) \equiv e^{\frac{i}{2}\varsigma^{\alpha}(x)\sigma_{\alpha}}\psi\,(0)\,, \tag{3}$$

cannot be defined. This happens when

1. D_J contains a *zero* of $\psi \equiv \xi_{\pm}$, $\eta_{\pm}$, $\varsigma^{\pm}$, or $\chi^{\pm}$;

2. ψ or $\mathbf{d}\psi$ is undefined somewhere in D_J, i.e. D_J contains a *singular point* of ψ;

3. the phases of each field in (3) are *defined,* but J pairs break away from PTC conjugacy. The transformations that create these states violate the *spin isometry condition*

$$\zeta^{\pm}\xi_{\mp} = 1 = \chi^{\pm}\eta_{\mp}. \tag{4}$$

4. J of the 4 gradients in $\mathcal{L}_g$ become *linearly* dependent in D_J, and so fail to span a 4-volume element. The remaining pairs span the $(4-J)$-surface over which the J broken out fields are quantized, as we show below.

We call the row spinors $\varsigma^{\mp}$ and $\chi^{\mp}$ in (1) the *Baryonic spinors.* They must be treated as *independent variables* from the *leptonic* (column) spinors $\xi_{\pm}$ and $\eta_{\pm}$ in the variation of $\mathcal{L}_g$ within each singular domain D_J. In the companion paper [7], we identify codimension $J = 1$, 2, 3, and 4 *topological defects* in the multi-spinor fields with leptons, bosons, hadrons and their reaction vertices, respectively. Inside the D_1, $\mathcal{L}_g$ gives Dirac equations coupling each chiral pair of matter fields through nonlinear scatterings with the vacuum fields, thus creating the effective masses of bispinor particles [7].

However, it is not necessary to unravel the detailed structure of these core regions to prove that they carry *integral charges—electric charge, lepton number,* and *baryon number*—and of *action*, provided that the "inner" solutions for $\mathcal{L}_g$ match the "outer" (*g.o.*) solutions for $\widehat{\mathcal{L}}_g$ outside the singular domains, i.e. in $\mathbb{M} \equiv \mathbb{M}_{\#} \setminus \cup D_J$.

Below we prove a $(3+1)$-dimensional *Clifford residue theorem* for Lie-algebra-valued forms, that says each singular domain contributes integral units of action and charge for *any* Lagrangian density that is a natural 4 form. The argument breaks down into four steps:

1. Separate the action into outer (field) and inner (matter) contributions,

$$S_g = \int_{\mathbb{M}} \widehat{\mathcal{L}}_g + \int_{\cup D_J} \mathcal{L}_g = S_F + S_M.$$

2. Show that the field action for the vacuum spin bundle $\hat{\Psi}$ over the compact base space, $\mathbb{M}_{\#} \equiv \mathbb{S}^1 \times \mathbb{S}^3$, is topologically quantized.

3. Act on $\hat{\Psi}$ with topologically nontrivial *active local* Einstein (E_A) transformations that may become singular in codimension-J domains D_J.

4. Show that the resulting *field actions* and *charges* are all topologically quantized over $\mathbb{M}$.

2 Spin Connections and Maurer-Cartan Forms

We briefly review how spinors factor the "internal" Lie-algebra $gl\,(2,\mathbb{C})$ of conformal spinors (see Appendix). The affine *spin connection* Ω gives the spin-space increment that corresponds to each space-time increment, and *vice versa.* Ω is a $gl\,(2,\mathbb{C})$-valued 1 form that enters into the covariant derivative to assure covariance under coupled internal/external spin transformations in any moving frame.

We specialize below to spacetime and spin frames adapted to a *Friedmann universe;* an expanding "3 brane" $S_3\,(T)$ that, at *"cosmic" time* T, is approximately a hypersphere $\mathbb{S}^3\,(a) \subset \mathbb{R}^4$, with radius

$$a\,(T) = e^{\frac{T}{a_\#}} a_\# \equiv \gamma a_\#. \tag{5}$$

Here $a_\#$ is the equilibrium radius [9]; γ is the conformal *scale factor.*

The *real* radial coordinate T is not directly visible to us as observers embedded in $S_3\,(T)$. In relativistic kinematics, T is replaced by *arctime* $x^0 \subset \mathbb{S}^1$: the arclength travelled on $\widetilde{\mathbb{S}}^3$ by a photon, projected down to $\mathbb{S}^3\,(a_\#)$, the fiducial three-sphere of stationery radius $a_\#$.

Arctime x^0 enters [9] as the real part of a *complex* time coordinate $z^0 \equiv x^0 + iy^0$; cosmic time $T \equiv y^0$ is the imaginary part. We do our local physics in a dilation-invariant way by projecting down to $\mathbb{M}_\# \equiv \mathbb{S}^1 \times \mathbb{S}^3\,(a_\#)$, Penrose's [4] conformal compactification of Minkowsky space, with canonical (Lie-algebra) "coordinates" $x = \left(x^0, x^1, x^2, x^3\right)$.

$\mathbb{M}_\#$ is a very nice space on which to work, because it is a Lie group:

$$\mathbb{M}_\# \equiv \mathbb{S}^1 \times \mathbb{S}^3 \sim U\,(1) \times SU\,(2)\,.$$

$\mathbb{S}^3$ has two natural representations of translation, Left (L) and Right (R), that derive from Left or Right translation in $SU\,(2)$. These are the two *chiralities.*

Adding a $u\,(1)$ generator σ_0 to each, we obtain $\sigma_\alpha \in u\,(1) \times su\,(2)_L$ and $\bar{\sigma}_\alpha \in u\,(1) \times su\,(2)_R$, the *left* and *right* Lie algebras. These must be viewed as *independent generators* of chiral $U\,(1) \times SU\,(2)$. However, note that $\bar{\sigma}_\alpha$ is the *dual* Lie algebra to $\sigma_\alpha \equiv (\sigma_0, \sigma_1, \sigma_2, \sigma_3)$, under the *Clifford-Killing form* for the Minkowsky metric, $\eta_{\alpha\beta} \equiv diag\,(1,-1,-1,-1)$:

$$\{\sigma_\alpha, \bar{\sigma}_\beta\} \equiv \sigma_\alpha \bar{\sigma}_\beta + \sigma_\beta \bar{\sigma}_\alpha = 2\eta_{\alpha\beta}\sigma_0. \tag{6}$$

$$\sigma^\alpha = \bar{\sigma}_\alpha; \quad \sigma^\rho \sigma_\rho = -2, \tag{7}$$

is the Lorenz-invariant form.

We may thus define the *Clifford product* of "spinorized" tangent vectors $a, b \in T\mathbb{M}_\#$,

$$\begin{gathered} a = a^\alpha \sigma_\alpha, \\ \bar{b} = b^\beta \bar{\sigma}_\beta : \tfrac{1}{2}\left(a\bar{b} + b\bar{a}\right) = \eta_{\alpha\beta} a^\alpha b^\beta \sigma_0 \equiv a_\beta b^\beta \sigma_{0.} \end{gathered} \tag{8}$$

This is the *scalar* σ_0 in the Lie algebra times the *Minkowsky* product of the vectors. Note that the Clifford scalar is picked *out* by the *Trace:*

$$\frac{1}{2}Tr\left(a\bar{b}\right) = a_\beta b^\beta = a_0 b^0 - a_1 b^1 - a_2 b^2 - a_3 b^3. \tag{9}$$

In curved spacetime (A11), the $\eta_{\alpha\beta}$ are replaced by the metric coefficients $g_{\alpha\beta}$.

The columns of spin frames (A5) are a basis for the fundamental L and R chirality spinors $\xi_\pm(x)$ and $\eta_\pm(x)$ painted on $\mathbb{M}_\#$ by the *spinorization maps*

$$\begin{array}{c} S : g_\pm(x) \equiv \exp\left(\frac{i}{2a_\#} x^\alpha \sigma^\pm_\alpha\right) : \mathbb{S}^1 \times \mathbb{S}^3 \longrightarrow U(1)_\pm \times SU(2)_L \\ \bar{S} : \bar{g}_\pm(x) \equiv \exp\left(\frac{i}{2a_\#} x^\alpha \bar{\sigma}^\pm_\alpha\right) : \mathbb{S}^1 \times \mathbb{S}^3 \longrightarrow U(1)_\pm \times SU(2)_R\,, \end{array} \tag{10}$$

where $\sigma^\pm_\alpha \equiv (\pm\sigma_0, \boldsymbol{\sigma})$. Their infinitesimal versions are the L- and R-invariant *Maurer-Cartan* 1 *forms*:

$$\begin{array}{c} TS(x) \equiv g_\pm^{-1} dg_\pm(x) = \frac{i}{2a_\#}\sigma^\pm_\alpha e^\alpha(x) : e_\beta(x) \longrightarrow \frac{i}{2a_\#}\sigma^\pm_\beta(x) \\ T\bar{S}(x) \equiv \bar{g}_\pm^{-1} d\bar{g}_\pm(x) = \frac{i}{2a_\#}\bar{\sigma}^\pm_\alpha \bar{e}^\alpha(x) : \bar{e}_\beta(x) \longrightarrow \frac{i}{2a_\#}\bar{\sigma}^\pm_\beta(x)\,. \end{array} \tag{11}$$

The Maurer-Cartan 1 forms give the images in the "internal" Lie algebras $u(1)_\pm \times su(2)_L$ and $u(1)_\pm \times su(2)_R$ of infinitesimal L and R translations on $\mathbb{M}_\#$; i.e. the canonical spin-space increments that accompany a spacetime translation on $\mathbb{M}_\#$.

In the presence of a *source,* a translation is accompanied by *active local* spin space increments $\ell(x)$ and $r(x)$ in the reference frame of an observer O. O then experiences the *vector potentials*

$$\begin{array}{c} \Omega_L \equiv \ell^{-1}\mathbf{d}\ell = \Omega_{L\alpha} e^\alpha; \\ \Omega_{L\alpha} = \ell^{-1}\partial_\alpha \ell; \qquad \Omega_R \equiv r^{-1}\mathbf{d}r. \end{array} \tag{12}$$

The Lie-algebra-valued 1 forms, or *spin connections* Ω_L and Ω_R are the *Maurer-Cartan* 1 *forms* for local $Gl(2,\mathbb{C})$ *deformations* $\ell(x)$ and $r(x)$ of the canonical maps (10) of spacetime into spin space (see (A10) below). *Regular g.o.* perturbations do not change the rank of the mapping ψ of physical space to spin space.

3 Vector Potentials from Active Local Spin Transformations

Active local (E_A) transformations represent both local dilation/boost flows and local $U(1) \times SU(2)$ phase flows in the *geometrical optics* (*g.o.*) regime. E_A transformations on the tetrads (A8), (A9) are presented as *complexified* chiral

$$U(1) \times SU(2) \stackrel{\mathbb{C}}{\Longrightarrow} GL(2,\mathbb{C})$$

spin transformations on the canonical spin frames:

$$\begin{array}{c}\ell(z) = \ell(0)\, L(z) \equiv \ell(0) \exp \frac{i}{2}\left(\theta_L^\alpha(z) + i\varphi_L^\alpha(z)\right)\sigma_\alpha \equiv \ell(0)\, e^{\frac{i}{2}\varsigma_L^\alpha(z)\sigma_\alpha} \\ \bar{r}(z) = \bar{R}(z)\, \bar{r}(0) \equiv \exp\left(\frac{i}{2}\left(\theta_R^\alpha(z) + i\varphi_R^\alpha(z)\right)\sigma_\alpha\right) \bar{r}(0) \equiv e^{\frac{i}{2}\varsigma_R^\alpha(z)\sigma_\alpha}\, \bar{r}(0)\,,\end{array} \tag{13}$$

where we may take $\ell(0) = \sigma_0 = r(0)$.

In a spin bundle E with a momentum flow $y_\beta(x)$, the Cartan moving spin frames (13) are *path dependent* functions of x. The $\theta_L^\alpha(x)$ are the coefficients of the *anti-Hermitian* (aH) matrices $\frac{i}{2}\sigma_\alpha$ that generate (local) *unitary* $U(1) \times SU(2)_L$ spin transformations. Their differentials are the *electroweak vector potentials:*

$$\frac{i}{2}\mathbf{d}\theta^\alpha \sigma_\alpha \equiv W_\beta e^\beta.$$

The $\varphi_L^\alpha(x)$ are the coefficients of the Hermitian (H) generators $\frac{1}{2}\sigma_\alpha$ which give the local dilation/boost flow, and whose differentials are the *gravitational potentials,*

$$\frac{i}{2}\mathbf{d}\varphi^\alpha \sigma_\alpha \equiv \Phi_\beta e^\beta.$$

For example, the Newtonian potential $\mathbf{d}\varphi^0(x)$ represents a local contraction of the spatial step corresponding to a fixed increment in the amplitude of the spinor fields. Outside the singular loci, we expect the phase flow to be *analytic,* so the Cauchy-Riemann equations will hold:

$$\frac{\partial \zeta^\alpha}{\partial \bar{z}^\beta} = 0 \Longrightarrow \frac{\partial \theta^\alpha}{\partial x^\beta} = \frac{\partial \varphi^\alpha}{\partial y^\beta}; \quad \frac{\partial \theta^\alpha}{\partial y^\beta} = -\frac{\partial \varphi^\alpha}{\partial x^\beta}. \tag{14}$$

There the $gl(2, \mathbb{C})$ phase factors $\theta^\alpha(z)$ and $\varphi^j(z)$ in (13) are functions of z, the position-momentum coordinates assigned to a point in phase space by an observer, O. E transformations thus act [9] on the *complexified* tetrads

$$\begin{array}{c} q_\alpha(z) \equiv \ell(z) \otimes_\alpha \bar{r}(z)\,; \\ z \equiv z^\beta \equiv x^\beta + iy^\beta,\ \beta = 0, 1, 2, 3; \\ z^\beta \in \mathbb{CM} \subset T^*\mathbb{M}. \end{array} \tag{15}$$

The z^β are 4 *complex* coordinates on the Dirac *phase space.*

Just as $q(t)$ is the complex position-momentum vector for the harmonic oscillator $q(t) = i\omega q(t)$ as a first order system, the q_α are complex vectors in the position-momentum frame bundle $\mathbb{CM}$. This complex structure, along with the antisymmetric inner product

$$\langle \ell_{\mathbf{1}}, \ell_{\mathbf{2}} \rangle \equiv |\ell| = \ell_{\mathbf{1}}^T \epsilon \ell_{\mathbf{2}} \equiv \ell^{\mathbf{1}} \ell_{\mathbf{2}}, \tag{16}$$

gives a *symplectic* structure [10] on $T^*\mathbb{M}$. The *norm* of a spin frame is its *determinant* (16), the area in phase space that it spans.

The canonical spin connections on $\mathbb{M}_{\#}$ are obtained for $y^{\beta}=0$; they are the Maurer-Cartan 1 forms (11) on the Lie groups $U(1)_{\pm}\times SU(2)_L$ and $U(1)_{\pm}\times SU(2)_R$:

$$\hat{\Omega}_{L\pm}=g_{\pm}^{-1}\mathbf{d}g_{\pm}=\frac{i}{2a_{\#}}\sigma_{\alpha}^{\pm}e^{\alpha},\qquad \hat{\Omega}_{R\pm}=\bar{g}_{\pm}^{-1}\mathbf{d}\bar{g}_{\pm}=\frac{i}{2a_{\#}}\bar{\sigma}_{\alpha}^{\pm}e^{\alpha}. \tag{17}$$

It is important to note that Lagrangian (1) contains wedge products of *right* and *left* Lie algebra-valued forms:

$$\begin{aligned}\hat{\Omega}_{+}^{L}\wedge\hat{\Omega}_{L}^{+}&=\tfrac{i}{2a_{\#}}\left(\sigma_0e^0+\bar{\sigma}_je^j\right)\wedge\tfrac{i}{2a_{\#}}\left(\sigma_0e^0+\sigma_ke^k\right)\\&=-\tfrac{1}{2a_{\#}^2}\sigma_j\left[e^0\wedge e^j+\tfrac{i}{2}\epsilon_{k\ell}^{j}e^k\wedge e^{\ell}\right],\end{aligned} \tag{18}$$

for the vacuum spin connections (17) on $\mathbb{M}_{\#}$. Note that $\Omega^L\wedge\Omega_L$ includes both *magnetic* $\left(e^k\wedge e^{\ell}\right)$ and *electric* $\left(e^0\wedge e^j\right)$ components. *No electric components would have appeared without the P-conjugation in (18).*

The wedge product of two left and two right Maurer-Cartan 1 forms makes the *Maurer-Cartan* 4 *form,* the *scalar* in the Lie algebra times the volume form:

$$\Omega^4\equiv\frac{1}{2}Tr\Omega^L\wedge\Omega_L\wedge\Omega^R\wedge\Omega_R; \tag{19}$$

$$\begin{aligned}\text{e.g.}\quad\hat{\Omega}^4&=\left(\tfrac{i}{2a_{\#}}\right)^4\tfrac{1}{2}Tr\sigma_0e^0\wedge\sigma_1e^1\wedge\bar{\sigma}_2e^2\wedge\bar{\sigma}_3e^3\\&=\tfrac{i}{16a_{\#}^4}\tfrac{1}{2}Tr\sigma_0e^0\wedge e^1\wedge e^2\wedge e^3\equiv\tfrac{i}{16a_{\#}^4}d^4V.\end{aligned} \tag{20}$$

The $\frac{1}{2}Tr$ picks out the *scalar* component, σ_0.

By definition, all integrands must be *scalars*, i.e. multiples of the Clifford unit, σ_0. This is especially clear in *curved space*, where the Clifford-algebra frame $\boldsymbol{\sigma}(z)$ varies from point to point. It is a standard calculation to check that $\int\Omega^4$ is invariant with respect to the full *conformal group* of nonsingular local E transformations, $e^{\alpha\prime}(z)=\Lambda_{\beta}^{\alpha}e^{\beta}(z)$, of the 1 forms and their "internal" representations (13), (A10) on spinor and spin-vector fields. $\int\Omega^4$ is also P, T, and C *invariant.* Furthermore, *scalar functions,* $f(x)\Omega^4$, *of the Maurer-Cartan* 4 *form are the only* 4 *forms that can be invariantly integrated!* This is because all natural 4 forms are scalar multiples of the volume form, (19).

Our Lagrangian density, Ω^4, of (1) is the invariant measure on the Einstein group

$$E=\mathbb{C}\left(U(1)\times SU(2)\right)^4=GL(2,\mathbb{C})^4. \tag{21}$$

Its integral gives the *covering number* W of the group manifold E over space-time, $\mathbb{M}$. Local extrema of the action integral (1) over $\mathbb{M}$ are achieved [7], [11] when *all* 4 pairs are PTC symmetric. From (19),

$$\begin{aligned}S_F\equiv\int_{\mathbb{M}}\mathcal{L}_g\xrightarrow{PTC}\tfrac{i}{2}\int_{\mathbb{M}}Tr\Omega^L\wedge\Omega_L\wedge\Omega^R\wedge\Omega_R\equiv\int_{\mathbb{M}}\Omega^4\\=-16\pi^3W.\end{aligned} \tag{22}$$

For $\mathbb{M}_{\#} = \mathbb{S}^1 \times \mathbb{S}^3$,

$$i \int_{\mathbb{S}^1 \times \mathbb{S}^3} \hat{\Omega}^4 \equiv -16\pi^3. \tag{23}$$

Spin frames (10) are the fundamental degree-1 maps of spin space over $\mathbb{M}_{\#}$.

When *singularities* of map (11) that assigns spin space increments to spacetime increments are present, we simply restrict TS to the *regular region* $\mathbb{M} \equiv \mathbb{M}_{\#} \backslash \cup D_J$ where all 4 spin connections are *defined.* The singular loci D_J are the supports of *matter fields* in this model.

The global spin connections $\hat{\Omega}$ provide a minimum vacuum energy (23). But they have another dramatic effect. When wedge products $\hat{\Omega}^{4-J}$ multiply local perturbations $\tilde{\Omega}^J$, they effectively quantize their Hodge *dual* fields over Poincaré *dual* cycles γ^{4-J}! This happens because products of Clifford-algebra-valued forms require both their Clifford and Hodge duals to make the Clifford scalar σ_0 times the volume element (20). This leads to a *residue theorem* below that classifies the topological obstructions D_J to relaxation of the field energy $V_F = -S_F$, to the global minimum $16\pi^3$ of (23).

4 Clifford Residues and *de*Rham Cohomology

The *charges* in nature—electric charge, mass, baryon number, etc.—are detected by integrating far fields in the regular region, *outside* the supports B_3 of their respective current 3 forms $*J(x)$. If the *same* far field could be produced by an active local spin transformation, $T_A(x) \in E_A$, acting on the vacuum fields around B_3, then the *same* charge would be detected within B_3. What happens inside our singular domains D_J is that the diagonal (PTC-symmetric) subalgebra breaks back to the full Lie algebra of *independent* $L \times R$ spin transformations:

$$gl(2,\mathbb{C})_{PTC} \xrightarrow{D_J} gl(2,\mathbb{C})_L \oplus gl(2,\mathbb{C})_R \tag{24}$$

for each of the $J = 1, 2, 3$, or 4 chiral pairs that break away from PTC conjugacy. We now *remove* open neighborhoods B_J containing each singular locus D_J, and consider the effect on action integral (22).

Uhlenbeck's theorem and Taubes patching [1] assure us that we can replace any vector potential singular inside a domain D_J by a *regular* connection, and change the action by an integral multiple of $8\pi^2$. We prove an analogous result for spin bundles below.

Suppose $\ell(z)$ of (15) is a section of the (left) $gl(2,\mathbb{C})$ spin-frame bundle over the Dirac phase space $\mathbb{CM} \subset \mathbb{C}^4$, with the singular loci removed [11]. We may write $\ell(z)$ in polar form as

$$\begin{aligned} \ell(z) &= \ell(0)\exp\left(\tfrac{i}{2}\theta_L^\alpha(z) - \tfrac{1}{2}\varphi_L^\alpha(z)\right)\sigma_\alpha \\ &\equiv \ell(0)\exp\tfrac{i}{2}\varsigma_L^\alpha(z)\,\sigma_\alpha, \end{aligned} \tag{25}$$

just as we may write a complex function $w(z)$ of one complex variable as

$$\begin{aligned} w(z) &= w(0)\exp\left(i\theta(z) - \varphi(z)\right) \\ &\equiv w(0)\exp i\varsigma(z), \end{aligned}$$

with the phase $\varsigma(z)$ *complex.*

When phase singularities are present, θ becomes path-dependent. But φ does not, provided $w(z)$ is single-valued. The phase advance around a 1-cycle γ parametrized by t, enclosing N zeroes and M poles of w, is the *logarithmic residue*

$$\int_\gamma w^{-1}(z)\,dw(z) \equiv \int_\gamma w^{-1}\left(\frac{dw}{dt}\right)dt = i\int_\gamma d\theta(z) = i2\pi m, \tag{26}$$

where $m \equiv N - M$. It detects the *winding number of the* $u(1)$ *phase about singularities* by integrating about 1-cycles that *lie completely within the regular region.*

The analog for spin bundles E is obtained by integrating $gl(2,\mathbb{C})$-valued m forms about cycles γ_m that *lie completely within the regular region* $\mathbb{CM}$. On γ_m, $z \equiv \left(z^0, z^1, z^2, z^3\right)$ is parameterized by t^α.

We define the integral of a $gl(2,\mathbb{C})$-valued m form ω^m on and m-chain γ_m as the integral of its *scalar component,*

$$\int_{\gamma_m}\omega^m \equiv \frac{1}{2}\int_{\gamma_m} Tr\omega^m.$$

Note (A11) that products of *left* and *right* $gl(2,\mathbb{C})$-valued forms make the Clifford scalar, σ_0. We call such products *Clifford-algebra-valued forms.*

We may now state and prove a *residue theorem* for Clifford-algebra-valued Forms (see [12], [13] for the $\gamma_m = S_{n-1}$ case):

Theorem 1: The *Clifford residues,*

$$\begin{aligned} \int_{\gamma_1}\ell^{-1}\mathbf{d}\ell &= i2\pi m, \\ \int_{\gamma_3} r^{-1}\mathbf{d}r \wedge \mathbf{d}r^{-1}r \wedge \ell^{-1}\mathbf{d}\ell &= 8\pi^2 m_1 m_2 m_3, \\ \text{and} \qquad \int_{\gamma_4}\mathbf{d}\ell^{-1}\ell \wedge r^{-1}\mathbf{d}r \wedge \mathbf{d}r^{-1}r \wedge \ell^{-1}\mathbf{d}\ell &= i16\pi^3 m m_1 m_2 m_3 \end{aligned} \tag{27}$$

for bundles of $gl(2,\mathbb{C})$ spin frames over $\mathbb{CM}$ are quantized about 1-cycles γ_1, 3-cycles γ_3, and 4-cycles γ_4. The periods m, m_1, m_2, and m_3 are integers that are invariant under nonsingular PTC-symmetric local *deformations,*

$$\begin{aligned} \ell'(z) &= \ell L(z), \\ r'(z) &= R(z)\,r = L^{-1}(z)\,r, \end{aligned} \tag{28}$$

provided that $\ell'(z)$ and $r'(z)$ remain single-valued about γ.

Proof: Using the identity

$$\exp\left(\frac{i\varsigma}{2}\hat{\varsigma}^{\alpha}\sigma_{\alpha}\right)=\cos\frac{\varsigma}{2}+i\left(\sin\frac{\varsigma}{2}\right)\hat{\varsigma}^{\alpha}\sigma_{\alpha}, \tag{29}$$

we calculate from (25) that

$$\begin{aligned}\ell^{-1}\mathbf{d}\ell\left(z\right)&=\left(\tfrac{i}{2}d\theta^{\alpha}\left(z\right)-\tfrac{1}{2}d\varphi^{\alpha}\left(z\right)\right)\sigma_{\alpha}\\ &\equiv\tfrac{i}{2}d\varsigma^{\alpha}\left(z\right)\sigma_{\alpha}.\end{aligned} \tag{30}$$

Assuming that the column spinors in $\ell\left(z\right)$ are single-valued on γ_1, the $U\left(1\right)$ phase advance, $\frac{1}{2}\triangle\theta^0$, about γ_1 must be an integral multiple of 2π:

$$\begin{aligned}\textstyle\int_{\gamma_1}\ell^{-1}\mathbf{d}\ell&=\textstyle\int_{\gamma_1}\frac{i}{2}d\varsigma^0\sigma_0=\int_{\gamma_1}\left(\frac{i}{2}d\theta^0-\frac{1}{2}d\varphi^0\right)\sigma_0\\ &=\tfrac{i}{2}\left[\triangle\theta^0\right]_{\gamma_1}-\tfrac{1}{2}\left[\triangle\varphi^0\right]_{\gamma_1}=i2\pi m.\end{aligned} \tag{31}$$

Integral (31) is the *period* about a homology 1-cycle, $\gamma_1\subset H_1\left(\mathbb{CM}\right)$, of the nonexact differential 1 form,

$$\Omega\equiv\ell^{-1}\mathbf{d}\ell\in H^1\left(\mathbb{CM}\right),$$

which belongs to the first deRham cohomology class of $\mathbb{CM}$. Its *period,* m, is invariant under both homologous deformations, $\gamma_1'\in H_1\left(\mathbb{M}\right)$, of the cycle, and nonsingular E perturbations, $\ell'\left(z\right)=\ell L\left(z\right)$, of the spin frame. If γ_1 is parametrized by time t, integral (31) measures the *difference* $\triangle\theta^0$ of the $U\left(1\right)$ phase shifts between paths—or the phase shift along a path that winds around the worldtube $D_3\times I$ of a massive particle. Integral (31) then gives the Bohr-Sommerfeld quantization conditions.

In the *spinfluid regime,* where the dilation/boost flow $y^{\alpha}=y^{\alpha}\left(x^{\alpha}\right)$ is a path-dependent function of 4 position x^{α}, we could choose the $x^{\alpha}\equiv t^{\alpha}$ as our integration parameters. Alternatively, we could choose spherical-polar coordinates and parametrize the spatial 3-ball D_3 by (r,θ,φ). D_3 compactifies to a 3-cycle γ_3 when the perturbed fields must match the vacuum fields on its boundary.

We get scalar-valued 3 forms in (27) from terms in $\sigma_r e^r\wedge\sigma_\theta e^\theta\wedge\sigma_\varphi e^\varphi=i\sigma_0 e^r\wedge e^\theta\wedge e^\varphi$. For example, suppose γ_3 contains a radially symmetric $SU\left(2\right)$ "hedgehog" monopole, i.e. a diagonal map from physical space $\mathbb{M}_{\#}\backslash 0$ to $\boldsymbol{\sigma}$-space:

$$\begin{aligned}\ell\left(x\right)&=e^{\frac{i}{2}\theta^0(x)\sigma_0}e^{\frac{i}{2}f(r)\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}}\\ r\left(x\right)&=e^{\frac{i}{2}\theta^0(x)\sigma_0}e^{\frac{i}{2}f(r)\hat{\mathbf{r}}\cdot\bar{\boldsymbol{\sigma}}}.\end{aligned} \tag{32}$$

Then

$$\begin{aligned}\textstyle\int_{\gamma_3}r^{-1}\mathbf{d}r\wedge\mathbf{d}r^{-1}r\wedge\ell^{-1}\mathbf{d}\ell&=-i\textstyle\int_{I(r)\times S_2(\theta,\varphi)}\mathbf{d}\left[f\left(r\right)\bar{\sigma}_r e^r\wedge\bar{\sigma}_\theta e^\theta\wedge\sigma_\varphi e^\varphi\right]\\ &=2\pi n\cdot4\pi m=8\pi^2 M,\end{aligned}$$

where $\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}\equiv\sigma_r$, and f has n radial cycles over $I\left(r\right)$.

More generally, the $SU(2)$ monopole may have angular dependence as well. Then

$$\begin{array}{c}\int_{\gamma_3} r^{-1}\mathbf{d}r \wedge \mathbf{d}r^{-1}r \wedge \ell^{-1}\mathbf{d}\ell = -i\int_{I(r)\times S_2(\theta,\varphi)} \mathbf{d}\left[f(r)\,\mathbf{g}(\theta,\varphi)\cdot\boldsymbol{\sigma}\right] \\ = 2\pi n\cdot 4\pi m = 8\pi^2 M,\end{array} \tag{33}$$

where M *must be an integer* for r to be single valued on spatial 2-surfaces $S_2(\theta,\varphi) = \partial D_3$ enclosing the support of the monopole fields.

Integral (33) is quantized because it is the *period* of the 3 form $\Omega^3 \equiv \frac{1}{2}Tr\Omega^{\wedge 3}$ over the 3-cycle γ_3. Similarly, the quantization of

$$\begin{array}{c}\int_{\gamma^4} \mathbf{d}\ell^{-1}\ell \wedge r^{-1}\mathbf{d}r \wedge \mathbf{d}r^{-1}r \wedge \ell^{-1}\mathbf{d}\ell \\ = (i2\pi n)\left(8\pi^2 m\right) = i16\pi^3 nm \equiv i16^3 N\end{array} \tag{34}$$

about 4-cycles $\gamma^4 = \gamma^1\times\gamma^3$ follows from the fact that only terms like

$$\sigma_0 e^0 \wedge \bar{\sigma}_1 e^1 \wedge \bar{\sigma}_2 e^2 \wedge \sigma_3 e^3 = i\sigma_0 d^4V$$

can make a scalar-valued 4 form. The integer N is the *action* contained in γ_4. ■

Heuristically, the reason for this quantization is easy to see: the $u(1)\times su(2)$ phase gradients of four independent spinor fields must be stretched over the four orthogonal spacetime directions in order for $\mathcal{L}_g$ of (1) to reproduce the 4-volume element. Integrals of these gradients are quantized over the "vacuum" $\mathbb{M} \equiv \mathbb{M}_{\#}\backslash\cup D_J$ and over localized E_A perturbations, provided that these patch smoothly into the vacuum phase distribution outside D_J. Such perturbations may add only integral units to the action. These integers are invariant under "small" E transformations (connected to the identity), and may change value by integer amounts only for the "large" E_A transformations associated with introducing another singularity.

On an expanding deformed space we may write our *topological Lagrangian* $\mathcal{L}_T$ in intrinsic coordinates as

$$\begin{array}{c}\mathcal{L}_T = \frac{i}{2}Tr\Omega^L \wedge \Omega_R \wedge \Omega^R \wedge \Omega_L \\ = \frac{i}{2}\left|-g\right|^{\frac{1}{2}} Tr\omega^L \wedge \omega_R \wedge \omega^R \wedge \omega_L.\end{array} \tag{35}$$

The $\omega \equiv \omega_\alpha E^\alpha$ are intrinsic spin-connection 1 forms in the coordinate frame of a co-moving observer and $\left|-g\right|^{\frac{1}{2}}$ is his 4-volume expansion factor:

$$e^0 \wedge e^1 \wedge e^2 \wedge e^3 = \left|-g\right|^{\frac{1}{2}} E^0 \wedge E^1 \wedge E^2 \wedge E^3.$$

Noting that PTC symmetry gives invariance of the trace,

$$R(x) = L^{-1}(x) \Longrightarrow \mathcal{L}'_T = TrL^{-1}(x)\Omega^4 L(x) = \mathcal{L}_T, \tag{36}$$

we have:

Corollary1: The action

$$S_T \equiv \int_{\mathbb{M}} \mathcal{L}_T \tag{37}$$

is invariant under the group E_P of *passive Einstein transformations* which connects tetrads and spinors that could represent the *same physical state* to observers using different external/internal coordinate/spin frames. These include the proper E transformations (A15), cosmic expansion, *plus* the P, T, and C reversals which preserve PTC symmetry:

$$\begin{array}{c} \left(q_\alpha, \xi_\pm, \chi^\pm\right) \xrightarrow{P} \left(\bar{q}_\alpha, \eta_\pm, \zeta^\pm\right) \\ \left(\xi_\pm, \eta_\pm\right) \xrightarrow{T} \left(\chi^\pm, \zeta^\pm\right), \qquad \left(\xi_\pm, \eta_\pm\right) \xrightarrow{C} \left(\xi_\mp, \eta_\mp\right); \end{array} \tag{38}$$

$$\left(\xi_\pm, \eta_\pm\right) \xrightarrow{PTC} \left(\zeta^\mp, \chi^\mp\right). \tag{39}$$

Furthermore, since *all* nonzero 4 forms are proportional to the volume element, with a local scale factor that may be taken up into $|-g|^{\frac{1}{2}}$ of (35), we have:

Corollary 2: Any Lagrangian density on the multiply-connected space $\mathbb{M} = \mathbb{M}_\# \backslash \cup D_J$ that is a *natural* (i.e.. E_P-invariant) 4 form must be locally proportional to $\mathcal{L}_T$ of (35).

Corollary 2 *apparently* relieves us mortals of the task of guessing the "real" grand-unified field Lagrangian, and gives us license to employ $\mathcal{L}_T$ as our Lagrangian density outside the singular loci. The problem is that we mortals apparently cannot experience a T reversed world, and so cannot know Ω^L and Ω^R of (35), nor the contribution to $|-g|^{\frac{1}{2}}$ from $\dot{y}^0 = \frac{\dot{a}}{a}$, the *rate* of cosmic expansion! The best we can do is to substitute (Ω_L, Ω_R) for $\left(\Omega^L, \Omega^R\right)$ and use the static approximation

$$\mathcal{L}_S = \frac{i}{2} Tr \Omega_L \wedge \Omega_L \wedge \Omega_R \wedge \Omega_R \tag{40}$$

as our Lagrangian density. The action integral

$$\begin{array}{c} S_S = \frac{i}{2} \int_{\mathbb{M}} Tr \Omega_0 e^0 \wedge \Omega_1 e^1 \wedge \Omega_2 e^2 \wedge \Omega_3 e^3 \\ = i \int_{\mathbb{M}'} \gamma^{-4} Tr \omega_0 E^0 \wedge \omega_1 E^1 \wedge \omega_2 E^2 \wedge \omega_3 E^3 \end{array} \tag{41}$$

may be done in either the extrinsic polar 1 forms $\Omega_\alpha e^\alpha$ on $\mathbb{M}$ or in the intrinsic 1 forms $\omega_\alpha E^\alpha$ on our dilated spacetime $\mathbb{M}'$. S_S is invariant with respect to *static* dilations $\gamma = \frac{a}{a_\#}$ (i.e. *scale* invariant) but cannot pick up $\dot{y}^0$, the *dilation rate.* S_S agrees with the topological action, S_T, in the T-symmetric (static) case.

5 Dual Residues and Charge Quantization

The global spin connections $\hat{\Omega}$, (17) provide the minimum vacuum energy (23). But they have another dramatic effect.

When wedge products $\hat{\Omega}^{4-J}$ multiply perturbations $\tilde{\Omega}^J$, they effectively quantize these over Poincaré *dual* cycles $*D^J \equiv B_{4-J}$. This happens because products of "polarized" Clifford-algebra-valued forms (17) require both their Clifford and Hodge duals to make the Clifford scalar σ_0 times the volume element—and so contribute to the action.

The vacuum fields can thus be used to "probe" inside the singular loci to produce new invariants—integrals of *Hodge dual fields over Poincaré-dual cycles.* These are the *charges.* We prove they are quantized below.

To account for the polarization of local J-fields form $\tilde{\Omega}^J(x)$ by the vacuum spin connections $\hat{\Omega}^{4-J}$, we write each spin connection as a perturbation $\tilde{\Omega}(x)$ added onto the global "vacuum" distribution $\hat{\Omega}$:

$$\Omega = \hat{\Omega} + \tilde{\Omega}.$$

The perturbed action $\tilde{S}_T$ will then have contributions from products of the $(4-J)$ vacuum connections, $J = 1$, 2, 3, or 4, and J perturbed fields inside each codimension—J singular domain.

If the perturbed fields $\tilde{\Omega}^J$ agree with the vacuum fields outside the singular domain,

$$*D^J \equiv B_{4-J} \subset \gamma_{4-J}, \tag{42}$$

then their contributions to the action must be *quantized* by Theorem 1.

The action contributed by each domain B_{4-J} is

$$\tilde{S}_T = \frac{i}{2}\int_{B_{4-J}\times I_J} Tr\tilde{\Omega}^J(x) \wedge \hat{\Omega}^{4-J} = -16\pi^3 m_J, \tag{43}$$

where I_J is a cycle parametrized by the J variables $*x$ orthogonal to B_{4-J}.

Now if the perturbed fields $\tilde{\Omega}(x)$ for $x \in B_{4-J}$ are *independent* of $*x$, the integral over I_J may be *factored out* of (43). The result is that the *dual* current $(4-J)$ forms $*\tilde{\Omega}^J$ become quantized over their supports B_{4-J}. Thus we have

Theorem 2: The *dual residues* $\int_{B_{4-J}} *\tilde{\Omega}^J \equiv Q_J$ are quantized, provided the *perturbed* fields agree (up to a trivial gauge transformation) with the *vacuum* fields outside a support B_{4-J}.

Proof: The proof is a calculation which we outline below for each case.

$J = 1$ **Case:** The 3 vacuum spin connections create the "polarized" 3-volume forms

$$\hat{\Omega}^3 = \frac{1}{16a_\#^3}\sigma_\alpha\epsilon^\alpha_{\beta\gamma\delta}e^\beta \wedge e^\gamma \wedge e^\delta \equiv \frac{2}{3}a_\#^3\sigma_\alpha * e^\alpha, \tag{44}$$

against which the 1 form field perturbations $\tilde{\Omega}(x) \equiv \tilde{\Omega}^{\beta}(x)\,\sigma_{\beta}$ are integrated. Each "vacuum polarization" (44) picks out its own internal direction σ_{α} in the *trace.*

The resulting contribution,

$$\tilde{S}_T \equiv \frac{2}{3} a_{\#}^{3} \int_{\gamma_1 \times B_3} \tilde{\Omega}^{\alpha}_{\ \alpha} e^{\alpha} \wedge * e^{\alpha} = -16\pi^3 M, \tag{45}$$

to the action is quantized: $M = \triangle W$ is an *integer,* if we require the perturbation to produce an integral change in the covering number W of internal (spin) space over external spacetime, $\mathbb{M}$.

When the perturbations $\tilde{\Omega}^{J}(\mathbf{x})$ are *time independent*, integrating (45) over $x^0 \in [0, 4\pi]$ gives the *quantized charge*

$$\int_{B_3} *J(\mathbf{x}) \equiv \int_{B_3} J_0(\mathbf{x})\, e^1 \wedge e^2 \wedge e^3 = 8\pi^2 Q. \tag{46}$$

The charge Q appears as the integral of the current 3 form $*J(x)$ *dual* to the 1 form perturbation (46),

$$\tilde{\Omega}(x) \equiv J(x) \equiv J_{\alpha}(x)\, e^{\alpha}, \tag{47}$$

produced when 1 chiral pair of "matter spinors" break away from PTC symmetry inside B_3. This creates a *bispinor Fermion.* The quantized charge (46) is then the Noether charge under *complex time translation,*

$$\begin{aligned} \textstyle\int_{B_3} *J = \int_{B_3} i\left(d\theta^0 + i d\varphi^0\right) e^1 \wedge e^2 \wedge e^3 \\ = 8\pi^2 (iq - m)\,. \end{aligned} \tag{48}$$

Elsewhere (Noether), we identify the real (external) part, m, of the time-translation charge with the *mass* and the imaginary (internal) part with the *electric charge* of a bispinor Fermion, in the $J = 1$ case.

Precisely the *dual* situation arises in the

$J = 3$ **Case:** Here integration against 1 vacuum connection $\hat{\Omega}$ quantizes the integrals of 3 forms around 1-cycles, or *orbits* γ_1. We identify the quantized integrals of 3-form densities in the $J = 3$ case as particle energy-momenta P. Integrating over γ_1 then gives the Bohr-Sommerfeld condition that quantizes the energy-momentum 1 forms of *particles* around orbits γ_1. This is the

$J = 4$ **Case:** The quantization of action

$$\int_{\gamma_1} \hat{\Omega} \int_{B_3} \tilde{\Omega}^3 \equiv \int_{\gamma_1} \left(P_0 e^0 - P_j e^j\right) = -16\pi^3 N. \tag{49}$$

We examine the most interesting case below, the

$J = 2$ **Case:** Here 2 forms become quantized over their *dual* 2 cycles. This gives quantization of *electric* flux—Gauss' law—after converting $\tilde{\Omega} \wedge \tilde{\Omega}$ to the *field* 2 form $K \equiv \mathbf{d}\tilde{\Omega} + \tilde{\Omega} \wedge \tilde{\Omega}$, then integrating by parts. ■

6 Chern Classes for Bispinor Bundles

Using expressions (A20) for the spin curvatures, we may rewrite the T-symmetric (T_S) part, (41), of the action as

$$\begin{array}{c} S_S = \frac{i}{2}\int_{\mathbb{M}} Tr\Omega_L \wedge \Omega_L \wedge \Omega_R \wedge \Omega_L \\ \equiv \frac{i}{2}\int_{\mathbb{M}} Tr\left(K_L - \mathbf{d}\Omega_L\right) \wedge \left(K_R - \mathbf{d}\Omega_R\right). \end{array} \tag{50}$$

Using the Bianchi identity $dK = K \wedge \Omega - \Omega \wedge K$, upon integration by parts (50) may be written as

$$S_S = \frac{i}{2}\int_{\mathbb{M}} TrK_L \wedge K_R + \int_{\mathbb{M}} Tr\Omega_L \wedge (K_L + K_R) \wedge \Omega_R + \sum M_J, \tag{51}$$

where the M_J are some Chern-Simons-type integrals about the boundaries of the singular domains. We showed [7], [11], that the term in $(K_L + K_R)$, the PT-symmetric (neutral) part of the net spin curvature, contains the *Palantini action for gravitation.* It vanishes in the PT antisymmetric (PT_A) case. There S_S is stationarized at

$$\hat{S}_A = \frac{i}{2}\int_{\mathbb{M}} TrK_L \wedge K_R \equiv -16\pi^3 C_2 \tag{52}$$

for the PT_A, $u(1) \times su(2)$ phase perturbations associated with *electroweak* potentials and charges.

C_2 is the *second Chern number* [14] *for the chiral bispinor bundle* $\psi : \mathbb{M} \longrightarrow L \oplus R$ under the Clifford-Killing form (6), (A11) for the *Minkowsky* metrics. This requires *wedge products of left and right Lie-algebra-valued 2 forms to make an E_P-invariant 4 form,* since the passive Einstein transformations include reciprocal Lorenz *boosts* on left and right spinors.

The chiral version of the second Chern form is thus the wedge *product* of the left-and-right $u(1) \times su(2)$-valued spin-curvature 2 forms,

$$\begin{array}{c} K_L \equiv \left(K^{\chi}_{L\beta}\right)\sigma_\chi e^\alpha \wedge e^\beta, \\ K_R \equiv \left(K^{\rho}_{R\delta}\right)\bar{\sigma}_\rho e^\gamma \wedge e^\delta. \end{array}$$

The PT_A part (52) of the action (50) is *quantized* because it is the second Chern number of a bispinor bundle. It resembles the Yang-Mills action $\int Tr\left(F \wedge *F\right)$, with Hodge $*$ replaced by P reversal. This resemblance is deeper than it appears, due to equation (A3).

The spin curvature 2 forms K_L and K_R *are* infinitesimal L and R spin transformations; they output infinitesimal $u(1) \times su(2)$ holonomy operators about the boundaries of their input two-cells. They thus naturally decompose into imaginary self-dual (left) and anti-self-dual (right) parts:

$$*K_L = iK_L \qquad *K_R = -iK_R, \tag{53}$$

since $\beta \longrightarrow i\beta$ takes us from $SO(4)$ in equations (A2), (A3) to $SO(1,3)$.

Furthermore, the unperturbed spin curvatures of canonical connections (17),

$$\begin{aligned}\hat{K}_{L\pm} &= -\tfrac{1}{a_{\#}^2}\sigma_j\left[\tfrac{i}{2}\epsilon^j_{k\ell}e^k\wedge e^\ell \pm e^0\wedge e^j\right]\\ &= (i\mathbf{B}_L+\mathbf{E}_L)\cdot\boldsymbol{\sigma}\\ \hat{K}_{R\pm} &= -\tfrac{1}{a_{\#}^2}\bar{\sigma}_j\left[\tfrac{i}{2}\epsilon^j_{k\ell}e^k\wedge e^\ell \mp e^0\wedge e^j\right]\\ &= (i\mathbf{B}_R+\mathbf{E}_R)\cdot\bar{\boldsymbol{\sigma}},\end{aligned}\tag{54}$$

are *chiral dual:*

$$\hat{K}_{R\mp} = i * \bar{\hat{K}}_{L\pm}.\tag{55}$$

The "vacuum fields" (54) are global *dyons,* with equal electric and magnetic fields distributed over $\mathbb{S}_3(a_{\#})$.

PT_A perturbations κ, for which $R^{\mp} = (L_{\pm})^{-1}$, preserve the metric tensor (A11), and therefore the Hodge $*$ operator. They thus preserve chiral duality conditions (55). For these, our $U(1)\times SU(2)$ action on $\mathbb{M}_{\#} = \mathbb{S}_1\times\mathbb{S}_3$ maps to an $\mathbb{R}\times SU(2)$ Yang-Mills action on $\mathbb{R}_4$:

$$\frac{i}{8}\int_{\mathbb{M}_{\#}} Tr\kappa_{L_\pm}\wedge\kappa_{R_\mp} \xrightarrow{PT_A} -\frac{1}{8}\int_{\mathbb{R}_4} Tr\kappa\wedge *\bar{\kappa}.\tag{56}$$

We may thus pull back the t'-Hooft/Jackiw-Noel-Rebbi multi-instanton solutions [14] on $\mathbb{R}_4$ to obtain localized *multi-dyon* solutions on $\mathbb{M}_{\#}$. The "global dyon" (54) centered at $0\in\mathbb{R}^4$ combines with a local dyon centered at $N\in\mathbb{S}_3(a_{\#})$, the north pole of our reference three sphere of radius $\lambda = a_{\#}$, to produce radial spin curvatures of:

$$\begin{aligned}\kappa_{L\pm} &= \left[\tfrac{2Ir^2}{(r^2+\lambda^2)^2}\right]\sigma_r\left[ie^\theta\wedge e^\varphi \pm e^0\wedge e^r\right],\\ \kappa_{R\pm} &= \left[\tfrac{2Ir^2}{(r^2+\lambda^2)^2}\right]\bar{\sigma}_r\left[ie^\theta\wedge e^\varphi \mp e^0\wedge e^r\right].\end{aligned}\tag{57}$$

Here we use spherical-polar coordinates

$$\begin{array}{cc}\sigma_r\equiv\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}; & \bar{\sigma}_r\equiv\hat{\mathbf{r}}\cdot\bar{\boldsymbol{\sigma}};\\ e^0\wedge e^r = dx^0\wedge dr, & e^\theta\wedge e^\varphi = r^2\sin\theta d\theta\wedge d\varphi\end{array}\tag{58}$$

in both physical space and spin space.

We suggest that the opposite nonAbelian magnetic fields in (57) could bind L- and R-chirality spinors into charged *bispinor Fermions.* Each contributes an action of

$$\frac{i}{2}\int_{\mathbb{M}_{\#}\backslash 0}\kappa_{L\pm}\wedge\kappa_{R\pm} = -16\pi^3 I^2 \equiv -16\pi^3 C_2\tag{59}$$

proportional to the *square* of the charge. But there is *also* a contribution from the interaction of each localized charge with the global fields (54)!

We show below how the interaction with the vacuum *magnetic* fields in (54) quantizes the flux of the *electric* field through any 2-surface that encloses a charge.

7 Topological Quantization of Electric Flux

To derive the quantization of electric flux from the quantization (59) of action, expand the PT_A parts of each spin curvature as the sum of the vacuum fields $\hat{\kappa}$ of (54) and the perturbations $\tilde{\kappa}$ due to local sources:

$$\begin{array}{l} \hat{\kappa}_{L\pm} + \tilde{\kappa}_{L\pm} \equiv \kappa^{\lambda}_{\alpha\beta}\sigma_{\lambda}e^{\alpha} \wedge e^{\beta} \\ \hat{\kappa}_{R\pm} + \tilde{\kappa}_{R\pm} \equiv \kappa^{\rho}_{\gamma\delta}\bar{\sigma}_{\rho}e^{\gamma} \wedge e^{\delta}. \end{array} \tag{60}$$

Substituting ansatz (60) into action (56), we obtain the cross terms

$$\mathcal{S}_c = \frac{-1}{2a_{\#}^2} \int \left[\tilde{\kappa}^{i}_{0j} + \epsilon^{k\ell}_{j}\tilde{\kappa}^{j}_{k\ell}\right]_L e^0 \wedge e^1 \wedge e^2 \wedge e^3 + P \tag{61}$$

between the local dyon fields and the vacuum fields (since only terms in $\sigma_j\bar{\sigma}_j$ will contribute to the trace).

In the PT_A case, the magnetic fields cancel, but the local *electric fields add*: We get a *charged* bispinor particle with a net "radial hedgehog" electric field:

$$\tilde{\kappa}^{\ j}_{Loj} + \tilde{\kappa}^{\ j}_{Roj} = \tilde{\kappa}^{\ j}_{oj}. \tag{62}$$

Inserting (62) into (61), we obtain the local $\wedge$ global interaction energy density

$$V_c = \frac{1}{2a_{\#}^2}\tilde{\kappa}^{\ j}_{0j} e^0 \wedge e^1 \wedge e^2 \wedge e^3. \tag{63}$$

Note that it is the *vacuum magnetic fields* in (54)—the spin curvatures of the canonical degree-1 maps (10) of $SU(2)$ over $\mathbb{S}^3(a_{\#})$—that endows potential energy to each "radial hedgehog" [15] configuration of *electric fields* $\tilde{\kappa}^{\ j}_{oj} e^0 \wedge e^j$ floating within it.

For example, suppose that the local electric field $E_3\left(x^1, x^2\right) = \tilde{\kappa}^{\ 3}_{03}\left(x^1, x^2\right)$ is in the 3 direction in both physical space and spin space, but its amplitude depends on the coordinates $\left(x^1, x^2\right)$ on a spatial 2 surface, S_{12}. We may then separate the PT_A part of action (61) into the product of integrals over S_{12} and over its normal coordinates $x^0 \in \mathbb{S}_1(a_{\#})$ and $x_3 \in \mathbb{S}_1(a_{\#})$:

$$\begin{array}{l} S_c = \frac{-1}{a_{\#}^2}\int_{S_{12}} E_3\left(x^1, x^2\right) e^1 \wedge e^2 \int_{\mathbb{S}_1(a_{\#})\times\mathbb{S}_1(a_{\#})} e^0 \wedge e^3 \\ \quad = -4\pi^2 \int_{S_{12}} E_3\left(x^1, x^2\right) e^1 \wedge e^2 = -16\pi^3 N. \end{array} \tag{64}$$

S_c is quantized over the normal surface S_{12} supporting the perturbation $E_3(x_1, x_2)$, via conditions (34), (52), (59). We have thus derived a version of Gauss' law

$$\int_{S_{12}} E_3\left(x^1, x^2\right) e^1 \wedge e^2 = 4\pi N,$$

where N is an *integer,* because the action (52) must change in *integral steps* $\triangle C_2$ for each localized "bubble" of field patched into the vacuum.

For a sphere $\mathbb{S}^2(\theta,\varphi)$ of radius r surrounding a charge with *radial* electric field $E_r(\theta,\varphi) = \tilde{\kappa}^r_{0r}(\theta,\varphi)$, (61) gives

$$\int_{\mathbb{S}^2(\theta,\varphi)} E_r(\theta,\varphi)\, r^2 \sin\theta d\theta \wedge d\varphi = 4\pi N. \tag{65}$$

More generally, (61) integrates the spin-space component of the field *normal* under spinorization map (11) to the spatial area element:

$$\int_{S^2} \mathbf{E}\cdot\mathbf{dA} = 4\pi N, \tag{66}$$

where S^2 is any 2-surface enclosing the charge. This is *Gauss' law.*

Quantization of the normal flux of the electric field over a closed *spatial* 2-surface thus follows directly from the quantization of the topological action (52). It is the *vacuum magnetic fields* in (54) that convert the integral of the *electric field* 2 form $\kappa_{0j}e^0 \wedge e^j$ into the integral of the *dual* 2 form $\kappa_{0j}e^\theta \wedge e^\varphi$ over a *spatial* homology cycle:

$$\int_{S^2} *\tilde{\kappa} = 4\pi N,$$

Gauss' law.

After accounting for the action of the homogeneous field (54), and the action (61) of localized charges immersed in this field, there is the remaining contribution of the product of 2 perturbed fields

$$S_K = \frac{i}{8}\int_{\mathbb{M}} Tr\tilde{\kappa}_L \wedge \tilde{\kappa}_R = -\frac{1}{8}\int_{\mathbb{M}} Tr\tilde{\kappa}_L \wedge *\tilde{\tilde{\kappa}}_L. \tag{67}$$

This is the Yang-Mills/Weinberg-Salaam "field action," which is usually added *by hand* to couple sources to their fields.

Note that the action (59) is *quadratic* in the charges, whether it comes from products (61) of the local field interacting with the global background field, with another charge, or with itself. This offers an explanation not only of *why* charge^2 has the units of action, but an estimate of the unit of charge^2 divided by the unit of action, i.e. of the *fine-structure constant,* α.

8 The Fine Structure Constant

Our spin connections, curvatures, and actions above have all been *geometrical* objects in the bundle of *spin frames* over $T^*\mathbb{M}$. No physical units like e or $\hbar$ have explicitly appeared. The topologically quantized electric charge and action appeared as "covering numbers" of internal space over external spacetime 2- and 4-cycles, respectively. However, since the relative increment α to the action introduced by adding a single charge in (64) is dimensionless, we might as well compute it in our geometric units.

The action S_c of (61), due to a unit electric charge immersed in the global magnetic field, is $16\pi^3$. As in (67) [16], we must multiply this by $\frac{1}{4}$ to obtain the Maxwell/Yang Mills *field* action produced by a single charge. If we take $\hbar$ as the *physical unit* of action, we obtain

$$\alpha \equiv \frac{e^2}{\hbar} = \frac{1^2}{4\pi^3} \approx \frac{1}{124} \tag{68}$$

as the number of units of action produced by adding a unit charge to the vacuum fields.

The value (68) does not agree very well with the observed value, $\alpha \approx \frac{1}{137}$. Either the mathematical model for charge quantization presented here fails to capture "real world" physics, or there is a "real world" correction to this model.

But expression (40), which we derived for the static (T-symmetric) case, *does* require a correction. When the radius $a\left(x^0\right)$ of our Friedmann universe $\mathbb{S}^3\left(a\left(x^0\right)\right)$ is expanding with Minkowsky time x^0, we need to include the factor $\dot{y}^0$ in the metric tensor. This shows up in $|g|^{\frac{1}{2}}$ of (35), but *not* in our static scale factor γ^{-4} of (41).

This correction arises because our intrinsic tetrads are co-moving with the Friedmann flow. We thus experience [16] a *Euclidean boost*—a tilt of our cotangent frame into the radial (y^0) direction:

$$\begin{gathered} \begin{bmatrix} dy^0 \\ |d\mathbf{x}| \end{bmatrix}' = \begin{bmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{bmatrix} \begin{bmatrix} dy^0 \\ |d\mathbf{x}| \end{bmatrix}, \\ \text{or} \quad \left(dy^0 + idx\right)' = e^{i\lambda}\left(dy^0 + idx\right), \\ \text{where} \quad \lambda \equiv \tan^{-1}\left(\tfrac{a_\#}{a}\mathring{a}\right) \equiv \tan^{-1}\dot{y}^0. \end{gathered} \tag{69}$$

The Minkowsky-time 1 form, $e^0 = dx^0 = |d\mathbf{x}| \equiv dx$, must suffer the same contraction as the spacelike increment to preserve $c = 1$, and special relativity. Thus our real Minkowsky 4-volume element V suffers the contraction

$$\begin{gathered} d^4V' \equiv \mathrm{Re}\left(e^0 \wedge e^1 \wedge e^2 \wedge e^3\right)' \\ = \cos 4\lambda\left(e^0 \wedge e^1 \wedge e^2 \wedge e^3\right) \equiv (\cos 4\lambda)\, d^4V \\ \Longrightarrow d^4V = (\cos 4\lambda)^{-1}\, d^4V', \end{gathered} \tag{70}$$

when projected to the static reference sphere $\mathbb{S}_3\left(a_\#\right)$.

If we, as co-moving observers, could somehow deduce the value $\lambda = \tan^{-1}\dot{y}^0$ of this radial tilt of our cotangent space, we could use (70) to correct our static approximation (41) for cosmic expansion. But we *can*, because the *spacelike*—or $SU(2)$—component of what we observe to be a *lightlike* translation changes when our spatial hypersurface is tilted with respect to the invariant null direction! It is precisely this tilt that gave [16] our correction to the Weinberg angle θ_W. This required a value of $\dot{y}^0 \approx 0.16$ to match the current best value of 28.5° for θ_W. Inserting $\dot{y}^0 = 0.16$ into (70), we obtain $(\cos 4\lambda)^{-1} \simeq 1.11$ as the correction factor (70) for our co-expanding 4-volume element. This gives a corrected action of

$$(1.11)\, 124 \simeq 124 + 13 = 137 \tag{71}$$

for a unit charge moving with the Friedmann flow. We can interpret the additional 13 units as its "kinetic energy" with respect to the stationery reference sphere $\mathbb{S}^3(a_\#)$.

From action (71), we obtain

$$\alpha' \approx \frac{1}{137}$$

for the fine structure constant, as measured by a co-moving observer. This is close to the measured value of $(137.037)^{-1}$.

9 Conclusions and Open Questions

From a class of Lagrangian densities which reduce to the Maurer-Cartan 4 form Ω^4 in the PTC-symmetric limit, we have derived the quantization of action and charge. These are simply the covering numbers of the internal phases in chiral spin bundles over 4-cycles and 2-cycles in the multiply-connected external spacetime $\mathbb{M} \equiv \mathbb{M}_\# \backslash \cup D_J$. It is *electric flux* that is quantized over spatial surfaces $S_2 = \partial D_3$ surrounding a charge, because the *vacuum magnetic fields* $\hat{\kappa}_{k\ell}$ convert the electric flux $\tilde{\kappa}_{0j}$ to quantized action. We thus have a realization of a "dual topological field theory" [15], [17], [18], in which Hodge star is replaced by a duality operation between internal Lie algebras. This is none other than the one induced by *Clifford* product (A11), in which the tetrads in (A10) *are* dyads in some fundamental, global spinor fields.

Thus, the metric tensor (A11) needed to contract two spin-1 tensors (vectors) is itself a spin-2 tensor. Any natural E_P-invariant 4 form—e.g. a Lagrangian density—*must* be the Clifford-scalar part of a spin-4 tensor,

$$\mathcal{L}_g \in \otimes^8 \subset \Lambda^4. \tag{72}$$

We have shown here that the simplest realization (1) of such a natural Lagrangian (72) gives quantized actions and charges.

When we add one unit of charge to the vacuum fields (54), we increment the action by ~ 137 units, as measured in our intrinsic frame, co-moving with cosmic expansion. We thus derive a value of $\alpha \sim (137)^{-1}$ for the fine structure constant.

Is this a numerical coincidence, or is there some relevance to fundamental physics in the mathematical structure we have developed here? More basically, do the cosmological background fields $\hat{K}$ really exist, and do they play a fundamental role in charge quantization? These questions await further investigation.

10 Appendix

Recall [9], [19], [20] that

$$\text{chiral } SO(4) \equiv \text{Spin } 4 \sim SU(2)_L \times SU(2)_R / \mathbb{Z}_2$$

presents a point $\left(a^0, \mathbf{a}\right) \in \mathbb{R}^4$ as the "quaternion," q, and its quaternionic conjugate, $\bar{q}$:

$$\begin{array}{c} q = a^0\sigma_0 + i\mathbf{a}\cdot\boldsymbol{\sigma} \equiv a^0\sigma_0 + ia^j\boldsymbol{\sigma}_j, \\ \bar{q} = a^0\sigma_0 + i\mathbf{a}\cdot\boldsymbol{\sigma} \equiv a^0\sigma_0 + ia^j\bar{\boldsymbol{\sigma}}_j; \\ j = 1, 2, 3. \end{array} \tag{A1}$$

The infinitesimal $so\,(4)$ isometries of $\mathbb{S}_3$,

$$\delta \begin{bmatrix} a^0 \\ \mathbf{a} \end{bmatrix} = \begin{bmatrix} 0 & -\boldsymbol{\beta}^T \\ \boldsymbol{\beta} & [\boldsymbol{\alpha}] \end{bmatrix} \begin{bmatrix} a^0 \\ \mathbf{a} \end{bmatrix}, \tag{A2}$$

are presented on the position quaternion, q, as

$$q' = Lq\bar{R} = e^{\frac{i}{2}(\boldsymbol{\alpha}+\boldsymbol{\beta})\cdot\boldsymbol{\sigma}} q \bar{e}^{\frac{i}{2}(\boldsymbol{\alpha}-\boldsymbol{\beta})\cdot\bar{\boldsymbol{\sigma}}}, \tag{A3}$$

with

$$\boldsymbol{\beta}^T \equiv (\beta_1, \beta_2, \beta_3)\,, \qquad \boldsymbol{\alpha}^T \equiv (\alpha_1, \alpha_2, \alpha_3)\,;$$

$$[\boldsymbol{\alpha}] \equiv \begin{bmatrix} 0 & \alpha_3 & -\alpha_2 \\ -\alpha_3 & 0 & \alpha_1 \\ \alpha_2 & -\alpha_1 & 0 \end{bmatrix}.$$

$\boldsymbol{\sigma}$ and $\bar{\boldsymbol{\sigma}}$ generate the *left* and *right* Lie algebras—which must be viewed as *completely independent* in chiral $so\,(4)$, giving 6 generators in all. Pure left-spin transformations $\boldsymbol{\alpha} = \boldsymbol{\beta}$ correspond to *self-dual* 2 forms, under the usual identification of skew-symmetric matrices $[\boldsymbol{\alpha}]$ with 2 forms [10]. Pure right transformations $\boldsymbol{\alpha} = -\boldsymbol{\beta}$ correspond to anti-self-dual 2 forms.

Dilations (e.g. of a Friedmann universe) may be included by adding a scalar generator σ_0; complexification of which gives an internal $U\,(1)$ phase shift. There are 4 representations, $\exp\left(\frac{i}{2}\theta^0\,(z) - \frac{1}{2}\varphi^0\,(z)\right)\sigma_0$, of translations in complex-time $z^0 \equiv x^0 + iy^0$, distinguished by the sign of the internal $u\,(1)$ phase advance with logradius y^0, $sgn\left(\frac{\partial\theta^0}{\partial y^0}\right)$, which we identify with the *charge* of the field, and by the *dilation behavior, sgn* $\left(\frac{\partial\varphi^0}{\partial x^0}\right)$, which distinguishes *leptonic* (light) from *baryonic* (heavy) spinors. These combine with the two chiralities to give **8** fundamental spinor representations [9] of the spin isometry group, or *Einstein group, E*; the globalization of the Poincaré group to a Friedmann universe. These make up the *Cartan moving spin frames*

$$\ell^{\pm}, u_{\pm}, r^{\pm}, v_{\pm}, \tag{A4}$$

pairwise. Each spin frame contains two basis spinors with opposite helicity: the fundamental null modes of the Dirac operators.

To match the standard convention for chiral bispinors on the conformal compactification [4] $\mathbb{M}_\# = \mathbb{S}^1 \times \mathbb{S}^3\,(a_\#)$ of Minkowsky space [9], we write the

leptonic spin frames $\ell^{\pm}(x)$ and $r^{\pm}(x)$ columnwise as the $GL(2,\mathbb{C})$ matrices

$$\begin{gathered}\ell^{\pm}(x) \equiv \left[\begin{array}{cc}\ell_{\mathbf{1}}(x) & \ell_{\mathbf{2}}(x)\end{array}\right]^{\pm} \\ = \sigma_0 \exp\left(\frac{i}{2a_{\#}}\left(\pm x^0\sigma_0 + x^j\sigma_j\right)\right) \equiv \sigma_0 g_{\pm}(x) \\ r^{\pm}(x) \equiv \left[\begin{array}{cc}r_{\dot{\mathbf{1}}}(x) & r_{\dot{\mathbf{2}}}(x)\end{array}\right]^{\pm} \\ = \bar{\sigma}_0 \exp\left(\frac{i}{2a_{\#}}\left(\pm x^0\bar{\sigma}_0 + x^j\bar{\sigma}_j\right)\right) \equiv \bar{\sigma}_0 \bar{g}_{\pm}(x).\end{gathered} \tag{A5}$$

We also write the *right* spin frame row-wise as

$$\begin{gathered}\bar{r}(x) \equiv \left[\begin{array}{c} r^{\dot{\mathbf{1}}}(x) \\ r^{\dot{\mathbf{2}}}(x)\end{array}\right] \equiv r^T(x)\,\epsilon^T, \\ \text{where } \epsilon \equiv i\sigma_2 = \left[\begin{array}{cc} 0 & 1 \\ -1 & 0\end{array}\right].\end{gathered} \tag{A6}$$

The overbar indicates space (P) reversal, or *Dirac conjugation*. We have the Lie-algebra isomorphism:

$$\bar{\sigma}_\alpha \sim \epsilon^{-1}(\sigma_\alpha)^T \epsilon = (\sigma_0, -\sigma_1, -\sigma_2, -\sigma_3). \tag{A7}$$

The *moving* spin frames $\ell(x)$ and $r(x)$ factor the *moving tetrads*. These are the spin-1 tensors

$$q_\alpha(x) = \sigma_{\alpha\,\dot{\mathbf{B}}}^{\;\mathbf{A}}\ell_{\mathbf{A}}(x) \otimes r^{\dot{\mathbf{B}}}(x) \equiv \ell \otimes_\alpha \bar{r}: \tag{A8}$$

$$\begin{gathered}q_0(x) \equiv \ell_{\mathbf{1}}(x) \otimes r^{\dot{\mathbf{2}}}(x) - \ell_{\mathbf{2}}(x) \otimes r^{\dot{\mathbf{1}}}(x) \equiv \ell \otimes_0 \bar{r}, \\ q_1(x) \equiv \ell_{\mathbf{1}}(x) \otimes r^{\dot{\mathbf{1}}}(x) + \ell_{\mathbf{2}}(x) \otimes r^{\dot{\mathbf{2}}}(x) \equiv \ell \otimes_1 \bar{r}, \\ q_2(x) \equiv i\left(\ell_{\mathbf{1}}(x) \otimes r^{\dot{\mathbf{1}}}(x) - \ell_{\mathbf{2}}(x) \otimes r^{\dot{\mathbf{2}}}(x)\right) \equiv \ell \otimes_2 \bar{r}, \\ q_3(x) \equiv \ell_{\mathbf{1}}(x) \otimes r^{\dot{\mathbf{2}}}(x) + \ell_{\mathbf{2}}(x) \otimes r^{\dot{\mathbf{1}}}(x) \equiv \ell \otimes_3 \bar{r};\end{gathered}$$

$$\bar{q}_\alpha(x) = r(x) \otimes_\alpha \bar{\ell}(x). \tag{A9}$$

The *matrix representations* $\mathbf{q}_\alpha(x)$ and $\bar{\mathbf{q}}_\alpha(x)$ of the moving tetrads $q_\alpha(x)$ and $\bar{q}_\alpha(x)$ have the matrix elements of the Pauli spin matrices σ_α and $\bar{\sigma}_\alpha$ with respect to the *moving* spin frames $\ell(x)$ and $r(x)$.

Under complex E transformations (13), the *matrix representations* of the tetrads with respect to the *original* basis $(\ell(0), \bar{r}(0))$ are,

$$\begin{gathered}\mathbf{q}'_\alpha(z) = L(z)\,\mathbf{q}_\alpha(0)\,\bar{R}(z) = e^{\frac{i}{2}\zeta_L^\beta(z)\sigma_\beta}\sigma_\alpha e^{\frac{i}{2}\zeta_R^\gamma(z)\sigma_\alpha} \\ \bar{\mathbf{q}}'_\alpha(z) = R(z)\,\mathbf{q}_\alpha(0)\,\bar{L}(z) = e^{\frac{i}{2}\zeta_R^\beta(z)\sigma_\beta}\bar{\sigma}_\alpha e^{\frac{i}{2}\zeta_L^\gamma(z)\sigma_\gamma},\end{gathered} \tag{A10}$$

where $\zeta^\alpha(z) \equiv \theta^\alpha(z) + i\varphi^\alpha(z)$. These obey the anti-commutation relations

$$\bar{\mathbf{q}}'_\alpha\mathbf{q}'_\beta + \bar{\mathbf{q}}'_\beta\mathbf{q}'_\alpha \equiv \left\{\bar{\mathbf{q}}'_\alpha, \mathbf{q}'_\beta\right\} = \left\{R\bar{\mathbf{q}}_\alpha\bar{L}, L\mathbf{q}_\beta\bar{R}\right\} = 2g_{\alpha\beta}\sigma_0 \tag{A11}$$

of the complexified Clifford algebra of (A10). The metric tensor in (A11) is *derived* from the tetrads (A8) and (A9)—which are in turn derived from the **8** fundamental global spinor fields, the dynamical variables in the theory.

L and R chirality spinors are coupled through the *Dirac operators*

$$\begin{aligned} D &\equiv iq^{\alpha}\partial_{\alpha} \\ \bar{D} &\equiv i\bar{q}^{\alpha}\bar{\partial}_{\alpha}. \end{aligned} \tag{A12}$$

These are the translation invariant derivations, or *Lie-algebra-valued vector fields* dual to the Maurer-Cartan forms (11).

Covariant derivatives ∇_{α} automatically appear in the Dirac operators (A12) by differentiating the Cartan moving spin frames in

$$\begin{aligned} \xi &\equiv \ell_A \xi^A \equiv \ell \boldsymbol{\xi} \\ \eta &\equiv r_{\dot{A}} \eta^{\dot{A}} \equiv r\boldsymbol{\eta}: \end{aligned}$$

$$\begin{aligned} \partial_{\alpha}\xi &\equiv \partial_{\alpha}\left(\ell\boldsymbol{\xi}\right) = \ell\partial_{\alpha}\boldsymbol{\xi} + \left(\partial_{\alpha}\ell\right)\boldsymbol{\xi} \\ &= \ell\left(\partial_{\alpha} + \Omega_{\alpha}\right)\boldsymbol{\xi} \equiv \ell\nabla_{\alpha}\boldsymbol{\xi}. \end{aligned}$$

The Dirac equations for a bispinor particle are [7], [3]:

$$\begin{aligned} D\boldsymbol{\xi} &\equiv iq^{\alpha}\left(\partial_{\alpha} + \Omega_{L\alpha}\right)\boldsymbol{\xi} = \tfrac{1}{2a_{\#}}\boldsymbol{\eta} \\ \bar{D}\boldsymbol{\eta} &\equiv i\bar{q}^{\alpha}\left(\bar{\partial}_{\alpha} + \Omega_{R\alpha}\right)\boldsymbol{\eta} = \tfrac{1}{2a_{\#}}\boldsymbol{\xi} \end{aligned} \tag{A13}$$

in the chiral representation [9].

To preserve Einstein covariance of the Dirac equations (A13), we must write all our matter fields with respect to the same moving spin frames that factor the spacetime tetrads (A8):

$$\begin{aligned} \xi_{\pm}(x) &\equiv \ell^{\pm}(x)\,\boldsymbol{\xi}_{\pm}(x) \equiv \ell^{\pm}(x)\left(\boldsymbol{\lambda}_{\pm} + \tilde{\boldsymbol{\xi}}_{\pm}(x)\right) \xrightarrow{g.o.} \tilde{\ell}^{\pm}(x)\,\boldsymbol{\lambda}_{\pm} \\ \eta_{\pm}(x) &\equiv r^{\pm}(x)\,\boldsymbol{\eta}_{\pm}(x) \equiv r^{\pm}(x)\left(\boldsymbol{\rho}_{\pm} + \tilde{\boldsymbol{\eta}}_{\pm}(x)\right) \xrightarrow{g.o.} \tilde{r}^{\pm}(x)\,\boldsymbol{\rho}_{\pm}. \end{aligned} \tag{A14}$$

$\boldsymbol{\lambda}_{\pm}$ and $\boldsymbol{\rho}_{\pm}$ are the homogeneous background, or *vacuum,* values of the "leptonic spinors," $\xi_{\pm}$ and $\eta_{\pm}$. $\tilde{\boldsymbol{\xi}}_{\pm}$ and $\tilde{\boldsymbol{\eta}}_{\pm}$ are their localized *envelope modulations.* These constitute electrons $\left(\tilde{\boldsymbol{\xi}}_{-} \oplus \tilde{\boldsymbol{\eta}}_{-}\right)$, positrons $\left(\tilde{\boldsymbol{\xi}}_{+} \oplus \tilde{\boldsymbol{\eta}}_{+}\right)$ and neutrinos $\left(\tilde{\boldsymbol{\xi}}_{+} \oplus \tilde{\boldsymbol{\eta}}_{-}\right)$ in this model [11]. The expressions $\xrightarrow{g.o.}$ hold in the *geometrical optics* (g.o.) regime where no two rays of the same spinor field cross; thus the phase advance (13) along paths is well-defined.

Constant $gl\left(2,\mathbb{C}\right)$ phase shifts generate the group of spacetime *isometries,* or *passive Einstein transformations,* E_P. These connect the spin frames that represent the *same state* to different observers:

$$\begin{array}{cc} \text{Spatial translations:} & \triangle\theta^j_L = \triangle\theta^j_R = \frac{\triangle x^j}{a_{\#}} \\ \text{Boosts:} & \triangle\varphi^j_L = \triangle\varphi^j_R = \frac{\triangle y^j}{a_{\#}} \\ \text{Arctime translations:} & \triangle\theta^0_L = -\triangle\,\theta^0_R = \pm\frac{\triangle x^0}{a_{\#}} \\ \text{Rotations:} & \triangle\theta^j_L = -\triangle\,\theta^j_R \end{array}. \tag{A15}$$

The *conformal dual* spinor to ξ_-,

$$\begin{aligned} \xi^- \equiv \xi_-^T \gamma\epsilon \equiv \boldsymbol{\xi}^- \ell_-, \\ \text{where} \quad \boldsymbol{\xi}^- \equiv \boldsymbol{\xi}_-^T \epsilon \quad \text{and} \quad \ell_- \equiv \epsilon^{-1} \left(\ell^-\right)^T \gamma\epsilon = \left(\ell_+\right)^{-1}, \end{aligned} \tag{A16}$$

is defined [21] so that E transformations (A15) along with

$$\text{Cosmic Expansion:} \quad \triangle\, \varphi_L^0 = \triangle\varphi_R^0 = \frac{\triangle y^0}{a_\#}, \tag{A17}$$

are *spin isometries* [11]. The E invariance of the $GL\left(2, \mathbb{C}\right)$ matrix product

$$\ell_- \ell^+ = \sigma_0 \tag{A18}$$

is what assures that the inner product $\xi^+ \xi_-$ is E invariant.

The *spin connections* (12) may thus be written as

$$\begin{aligned} \Omega_{L\pm} = \tilde{\ell}_\mp \mathbf{d} \tilde{\ell}^\pm \qquad \Omega_{R\pm} = \tilde{r}_\mp \mathbf{d} \tilde{r}^\pm; \\ \Omega^{L\pm} = \left(\mathbf{d}\tilde{\ell}_\pm\right) \tilde{\ell}^\mp \\ \Omega^{R\pm} = \left(\mathbf{d}\tilde{r}_\pm\right) \tilde{r}^\mp. \end{aligned} \tag{A19}$$

In *curved* spacetime, where $\mathbf{dd} \neq 0$, these possess *spin curvatures*

$$\begin{aligned} \tilde{\ell}_\mp \mathbf{dd} \tilde{\ell}^\pm = K_L^\pm = \left(\mathbf{d}\tilde{\Omega}_L + \tilde{\Omega}_L \wedge \tilde{\Omega}_L\right)^\pm \\ \tilde{r}_\mp \mathbf{dd} \tilde{r}^\pm = K_R^\pm = \left(\mathbf{d}\tilde{\Omega}_R + \tilde{\Omega}_R \wedge \tilde{\Omega}_R\right)^\pm. \end{aligned} \tag{A20}$$

Effective spin connections (A19) and curvatures (A20) appear in the *g.o.* regime for each *PTC-symmetric* pair of spinor fields in our Lagrangian 4 form (1). It is products of terms like these that give the topological forms (35), (52) for the action in the *PTC*-symmetric regime.

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